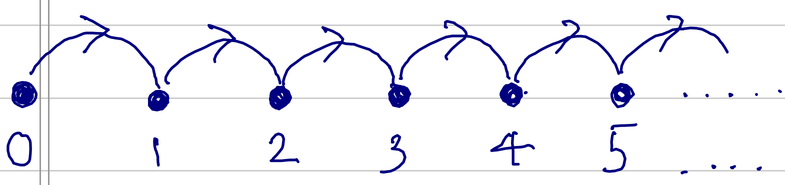
Chapter 2. Mathematical Induction

2.1. Weak Induction.

To prove a statement S for all natural number n we can do the following 2 steps.

① The Initial Step. Prove that S is true for the smallest value m for which it is defined.

② The Induction Step. Prove that if S is true for n, then it is also true for $n+1$.



Example 2.1. For all positive integers n

(a) $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(b) $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Solution:

(a) The initial step. When $n=1$, then the both sides of the identity are 1. Then (a) is true for $n=1$.

The induction step. Assume that (a) is true for n, i.e.

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad (\star)$$

then we need to show that (a) is also true for $n+1$, i.e.

$$1^2 + 2^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}, \quad (\star\star).$$

Indeed, write the LHS of $(\star\star)$ as

$$(1^2 + 2^2 + \cdots + n^2) + (n+1)^2.$$

By $(\star)$, we get

$$\begin{aligned}
1^2 + 2^2 + \cdots + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{(n+1)[(2n^2+n) + 6(n+1)]}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

$\square$

(b) Exercise.

Example 2.2.* Let $f(m)$ be the maximum number of domain into which m straight line can divide the plane. Then for all positive integer m, the equailty

$$f(m) = \frac{m(m+1)}{2} + 1 \quad \text{holds.}$$

Proof. It is easy to verify the equality for $n=1$. We have now only one line, and two domains. It is clear that $2 = \frac{1(1+1)}{2} + 1$.

For induction step, we assume that our equality holds for m, i.e any m lines divide the plane at most $\frac{m(m+1)}{2}+1$ domain, we need to show that any $m+1$ lines divide the plane into at most $\frac{(m+1)(m+2)}{2}+1$ domains.

Assume that we have $m+1$ lines on the plane, namely $l_1, l_2, \dots, l_m, l_{m+1}$. By the induction hypothesis $l_1, \dots, l_m$ divide the plane into at most $\frac{m(m+1)}{2}+1$ domains. Assume that l_{m+1} intersects k lines $t_1, \dots, t_k$ among $l_1, \dots, l_m$. It means that l_{m+1} passes $k+1$ domains formed by the other m lines, and cut each of them into two new domains. at most.

This means that l_{m+1} increases the number of domains by $k+1 \le m+1$. Therefore we have just proved that $f(m+1) \leq f(m) + m + 1$.

The equality holds if l_{m+1} intersect all m lines $l_1, l_2, \dots, l_m$ and l_{m+1} does not pass the intersections of these lines. Thus

$$f(m+1) = f(m) + m + 1 = \frac{m(m+1)}{2} + 1 + m + 1$$
$$= \frac{(m+1)(m+2)}{2} + 1. \quad \Box$$

Example 2.3. Prove that

$$\sum_{i=1}^{n} (2i-1) = n^2$$

Example 2.4. Prove that

a) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

b) $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

c) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2)$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

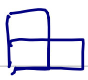
d*) Find the sum

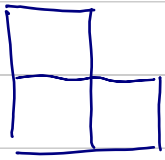
$$1 \cdot 2 \cdots K + 2 \cdot 3 \cdots (K+1) + \cdots + n(n+1) \cdots (n+K-1).$$

Example 2.5. Prove the identity

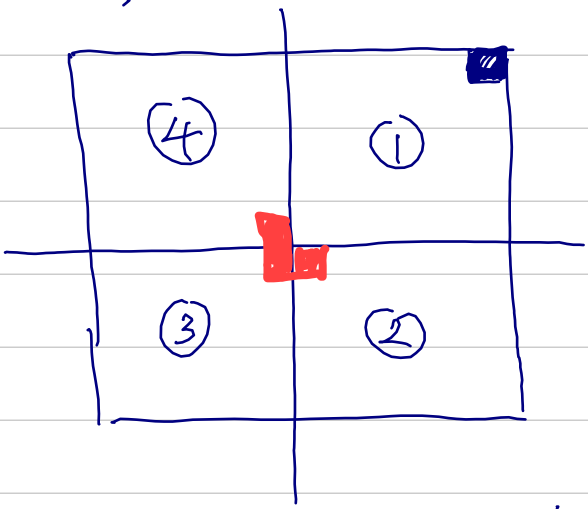
$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + \cdots + \frac{2^n}{1+x^{2^n}}$$

$$= \frac{1}{x-1} + \frac{2^{n+1}}{1-x^{2^{n+1}}}.$$

Example 2.6. Prove that the $2^n \times 2^n$ chess-board when the a corner cell removed can be covered by pieces of shape  with no gaps or overlaps.

Proof. $\underline{n=1}$.  is trivial

Assume that the statement is true for n, we need to show it is also true for $n+1$.
Consider $2^{n+1} \times 2^{n+1}$ chessboard with a corner, say upper-right, removed. Divide it into 4 part by the symmetry axes.



By induction part 1 can be tiled by . Next, remove a piece from the center. Then the remaining parts 2, 3, 4 can be tiled by (induction hypothesis). Thus our big chess board can be tiled by . $\square$

Example 2.7. Prove that for any positive integer $n > 1$:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{13}{24}$$

Proof: Let $S_n = LHS$.
Since $S_2 = \frac{7}{12} = \frac{14}{24} > \frac{13}{24}$, our inequality

is true for $n=2$.

② Suppose that $S_k > \frac{13}{24}$, we need to show that $S_{k+1} > \frac{13}{24}$.

$$S_{k+1} - S_k = \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$$

$$= \frac{1}{2(k+1)(2k+1)} > 0$$

$\Rightarrow$ $S_{k+1} > S_k > \frac{13}{24}$. $\square$

Example 2.8. Prove that

$$(1+\alpha)^n > 1 + n\alpha$$

where $\alpha > -1$, $\alpha \neq 0$, n is positive integer greater than 1.

Example 2.9. Prove that for any positive integer $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Example 2.10*. Prove that with positive $a_1, \dots, a_n$

$$\sqrt[n]{a_1 a_2 \dots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

(AM − GM inequality).

Proof. ① If $n=2$, our inequality becomes

$$\sqrt{a_1 a_2} \leq \frac{a_1+a_2}{2} \iff (\sqrt{a_1}-\sqrt{a_2})^2 \geq 0.$$

② Assume that if AM-GM is true for $n=k$, it is also true for $n=2k$.

$$\sqrt[2k]{a_1 \cdots a_{2k}} = \sqrt{\sqrt[k]{a_1 \cdots a_k}\,\sqrt[k]{a_{k+1} \cdots a_{2k}}}$$

$$\leq \frac{\sqrt[k]{a_1 \cdots a_k} + \sqrt[k]{a_{k+1} \cdots a_{2k}}}{2}$$

$$\leq \frac{(a_1+\cdots+a_k)/k + (a_{k+1}+\cdots+a_{2k})/k}{2}$$

$$= \frac{a_1+\cdots+a_{2k}}{2k}.$$

③ We will show that if AM-GM is true for $n=k>1$, it is also true for $n=k-1$.

Let $a_1, \dots, a_{k-1}$ be positive, and some $\lambda > 0$.

$$\sqrt[k]{a_1 \cdots a_{k-1}\lambda} \leq \frac{a_1+\cdots+a_{k-1}+\lambda}{k}. \quad (\star)$$

We will choose λ such that

$$\frac{a_1+\cdots+a_{k-1}+\lambda}{k} = \frac{a_1+\cdots+a_{k-1}}{k-1}$$

i.e.

$$\lambda = \frac{a_1+\cdots+a_{k-1}}{k-1}.$$

Thus $(\star)$ becomes

$$\sqrt[k]{a_1 \cdots a_{n-1} \frac{(a_1 + \cdots + a_{k-1})}{k-1}} \leq \frac{a_1 + \cdots + a_{k-1}}{k-1}$$

$$\Leftrightarrow \sqrt[k-1]{a_1 \cdots a_{k-1}} \leq \frac{a_1 + a_2 + \cdots + a_{k-1}}{k-1}. \qquad \square$$

Example 2.11. For all positive integers n, the number of all sub set of $\{1, 2, \ldots, n\}$ is 2^n.

Proof. ① The statement is obviously true for $n=1$. The set $\{1\}$ has $2 = 2^1$ subset ϕ and $\{1\}$.

② Assume that $\{1, \ldots, k\}$ has 2^k subsets, we will show that $\{1, \ldots, k+1\}$ has 2^{k+1} subsets.

We classify the subsets of $\{1, \ldots, k+1\}$ into two types.

(i) Subsets that do not contain $k+1$.
(ii) Subsets containing $k+1$.

The elements in the first group are exactly subsets of $\{1, \ldots, k\}$, and there are 2^k of them.
It is easy to construct a 1-1 correspondence between collections of subset of $\{1, \ldots, k\}$ and the subset of type (ii), say by adding the element $k+1$ to each subset of $\{1, \ldots, k\}$. Thus there are also 2^k subsets of type (ii).

In total, there are $2^k + 2^k = 2^{k+1}$ subsets of $\{1, 2, \ldots, k+1\}$.

$\square$

Example 2.12. Let n be a positive integer. n circles are given in the plane. They divide the plane into parts. Show that we can color the plane by two colors, so that no parts sharing boundary have the same color.

2.2. Strong Induction.

Stron induction has also 2 steps.

① The Initial Step. Prove the statement for the smallest value of n (or even several smallest values of n if needed).

② The Induction Step. Prove that if the statement is true for all integers less than $n+1$, it follows that the statement is also true for $n+1$.

Example 2.13. Let the sequence $\{a_n\}$ be defined by the relations $a_0 = 0$, $a_{n+1} = a_0 + a_1 + a_2 + \cdots + a_n + n + 1$ if $n \geq 0$. Prove that for all positive integers n, the equality $a_n = 2^n - 1$ holds.

Proof.

As $a_0 = 0$, the initial case is true.

For the induction step, assume that $a_i = 2^i - 1$ for all $i \leq n$, we need to show that $a_{n+1} = 2^{n+1} - 1$.

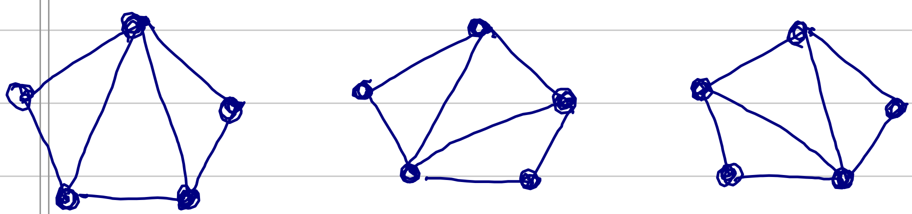
We have

$$a_{n+1} = a_0 + \cdots + a_n + n + 1$$

$$= (2^0 - 1) + \cdots + (2^n - 1) + n + 1$$

$$= 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1. \quad \square$$

Example 2.14.* A "triangulation" of a (convex) polygon is a way to divide the polygon into triangles by its non-intersecting diagonals. For example, we have here three triangulations of a pentagon:



Prove that the total number of triangulations of an $(n+2)$-gon is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The number C_n is called the nth **Catalan number.**

Hint. $C_n = \sum_{i+j=n+2} C_i C_j$.

Example 2.15. Prove that the total ways to cover a $2\times n$ rectangle by dominoes s.t. there are no gaps and overlaps is exactly F_n, where F_n is the nth Fibonacci number.

Example 2.16. Let $f(0)=1, f(1)=2$, and

$$f(n+1) = f(n-1) + 2f(n) \text{ if } n \geq 1.$$

Prove that $f(n) \leq 3^n$.

Example 2.17.

Any positive integer N can be expressed as a sum of distinct Fibonacci numbers

$$N = \sum_{j=1}^{m} F_{i_j+1} \qquad |i_j - i_{j-1}| \geq 1. \ (\star)$$

Proof. If N is a Fibonacci number, then the theorem is trivial.

For N is small, we check it by inspection.

Assum that the theorem is true for all integers up to and including F_n, and let $F_{n+1} \geq N > F_n$.

Now

$$N = F_n + (N - F_n), \text{ and } N \leq F_{n+1} < 2F_n$$

i.e., $N - F_n < F_n$. Thus $N - F_n$ can be written in the form

$$N - F_n = F_{t_1} + \cdots + F_{t_r}$$

$(t_{i+1} \leq t_i - 1,\ t_r \geq 2)$; and

$$N = F_n + F_{t_1} + \cdots + F_{t_r}. \qquad \square$$

Note: This can be considered as a weak form of **Zeckendorf's Theorem** stating that the expression (★) is unique and we can strengthen that $|i_i - i_{i-1}| \geq 2$.

Example 2.18. $n \geq 2$ points are selected along a circle and colored by red or blue. Prove that there are at most $\left\lfloor \frac{3n+4}{2} \right\rfloor$ non-intersecting chords which join 2 points of different colors.

Proof: we prove a stronger bound, $\left\lfloor \frac{3n-4}{2} \right\rfloor$

The result is obviously true for $n=2$.

Suppose we have already verified the theorem for all $k<n$. Draw any diagonal connecting two points of different colors (if exists). The circle is split into two parts. One has k points and the other $n-k$ points (including two end points of the chords, say A and B).

By induction the two parts has at most $\left\lfloor \frac{3k-4}{2} \right\rfloor$ and $\left\lfloor \frac{3(n+2-k)-4}{2} \right\rfloor$ chords

However each of them must contain the chord AB. So this chord is counted twice $\Rightarrow$ we have at most

$$\left\lfloor \frac{3k-4}{2} \right\rfloor + \left\lfloor \frac{3(n+2-k)-4}{2} \right\rfloor - 1$$

$$\leq \left\lfloor \frac{3(n+2)-8-2}{2} \right\rfloor = \left\lfloor \frac{3n-4}{2} \right\rfloor \qquad \square$$

Example 2.19. The sequence a_n is defined as follows: $a_0 = 9$, $a_{n+1} = 3a_n^4 + 4a_n^3$, $n > 0$.

Show that

a_n contain at least 2^n number nines in its decimal notation.

Example 2.20 Prove that

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$